Extending Gottesman Types Beyond the Clifford Group

Programming Languages for Quantum Computing, 2021

Robert Rand, Aarthi Sundaram, Kartik Singhal and Brad Lackey

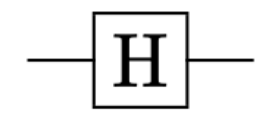
What types?

What types?

Daniel Gottesman, *The Heisenberg Interpretation of Quantum Computing*

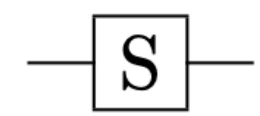
H

$$X \to Z$$
$$Z \to X$$



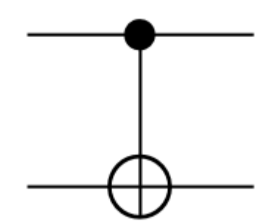
S

$$X \to Y$$
$$Z \to Z$$



CNOT

$$X \otimes I \to X \otimes X$$
$$I \otimes X \to I \otimes X$$
$$Z \otimes I \to Z \otimes I$$
$$I \otimes Z \to Z \otimes Z$$



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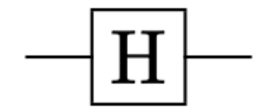
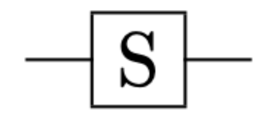
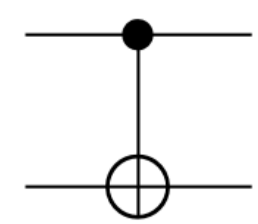
H	$X \to Z$ $Z \to X$	
S	$X \to Y$ $Z \to Z$	
CNOT	$X \otimes I \to X \otimes X$ $I \otimes X \to I \otimes X$ $Z \otimes I \to Z \otimes I$ $I \otimes Z \to Z \otimes Z$	

Rand, Sundaram, Singhal, Lackey, *Gottesman Types for Quantum Programs*

$H : X \to Z$ means H takes a ± 1 eigenstate of X (i.e. $|+\rangle, |-\rangle$) to a $+1$ eigenstate of Z (i.e. $|0\rangle, |1\rangle$)

Too Many Types?

$$H : X \to Z$$
$$H : Z \to X$$
$$S : X \to Y$$
$$S : Z \to Z$$
$$CNOT : X \otimes I \to X \otimes X$$
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Too Many Types?

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$$CNOT : X \otimes I \rightarrow X \otimes X$$
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$$CNOT : Z \otimes I \rightarrow Z \otimes I$$
$$CNOT : I \otimes Z \rightarrow Z \otimes Z$$

Intersection Types!

$$H : (X \rightarrow Z) \cap (Z \rightarrow X)$$
$$S : (X \rightarrow Y) \cap (Z \rightarrow Z)$$
$$CNOT : (X \otimes I \rightarrow X \otimes X) \cap (I \otimes X \rightarrow I \otimes X) \cap (Z \otimes I \rightarrow Z \otimes I) \cap (I \otimes Z \rightarrow Z \otimes Z)$$

Inspecting CNOT

$$CNOT : (Z \otimes I \to Z \otimes I) \cap (I \otimes Z \to Z \otimes Z)$$

Inspecting CNOT

$|0\rangle \otimes |\psi_1\rangle$

$$CNOT : (Z \otimes I \rightarrow Z \otimes I) \cap (I \otimes Z \rightarrow Z \otimes Z)$$

Inspecting CNOT

separable!

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Inspecting CNOT

separable!

$|0\rangle \otimes |\psi_1\rangle \quad |0\rangle \otimes |\psi_2\rangle \quad |\psi_3\rangle \otimes |0\rangle$

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Inspecting CNOT

separable!

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Inspecting CNOT

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$$\frac{CNOT : (Z \otimes I \rightarrow Z \otimes I) \cap (I \otimes Z \rightarrow Z \otimes Z)}{CNOT : (Z_1 \rightarrow Z_1) \cap (Z_2 \rightarrow Z \otimes Z)}$$

Inspecting CNOT

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$$\frac{\dfrac{CNOT : (Z \otimes I \rightarrow Z \otimes I) \cap (I \otimes Z \rightarrow Z \otimes Z)}{CNOT : (Z_1 \rightarrow Z_1) \cap (Z_2 \rightarrow Z \otimes Z)}}{CNOT : (Z_1 \cap Z_2) \rightarrow (Z_1 \cap Z \otimes Z)}$$

Inspecting CNOT

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CNOT behaves classically on classical inputs!

Inspecting CNOT

$CNOT : X_1 \to X \otimes X$

$CNOT : Z_2 \to Z \otimes Z$

Inspecting CNOT

$$\frac{CNOT : X_1 \to X \otimes X \qquad CNOT : Z_2 \to Z \otimes Z}{CNOT : (X_1 \to X \otimes X) \cap (Z_2 \to Z \otimes Z)}$$

Inspecting CNOT

$$\dfrac{\dfrac{CNOT : X_1 \to X \otimes X \qquad\qquad CNOT : Z_2 \to Z \otimes Z}{CNOT : (X_1 \to X \otimes X) \cap (Z_2 \to Z \otimes Z)}}{CNOT : (X_1 \cap Z_2) \to (X \otimes X \cap Z \otimes Z)}$$

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Bell pair

What's Missing?

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- Normal forms for types
  - Important for separability, measurement

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  - Important for separability, measurement
- Types for measurement
- Universality!
  - How do we deal with T?

Normalization

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Goal: A “row echelon form” for types.

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$$
\begin{aligned}
& X \otimes \ldots \\
\cap \;\; & I \otimes Z \ldots \\
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\cap \;\; & I \otimes I \otimes I \otimes X \ldots
\end{aligned}
$$

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$$

Rule:

$$\frac{|\psi\rangle : A \cap B}{|\psi\rangle : A \cap AB}$$

Normalization

Example

$$
\begin{aligned}
& Z \otimes Z \otimes I \\
\cap\ & Z \otimes Z \otimes Z \\
\cap\ & X \otimes X \otimes I
\end{aligned}
$$

Normalization

Example

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Normalization

Example

$$X \otimes X \otimes I$$
$$\cap \; Z \otimes Z \otimes I$$
$$\cap ZZ \otimes ZZ \otimes I \, Z$$

Normalization

Example

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\begin{aligned}
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\cap\ & I \otimes I \otimes Z
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Normalization

Example

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Normalization

Example

$$
\begin{aligned}
& X \otimes X \otimes I \\
\cap \; & Z \otimes Z \otimes I \\
\cap \; & \boxed{I \otimes I \otimes Z} - Z_3
\end{aligned}
$$

Normalization

Example

Bell

$$X \otimes X \otimes I$$

$$\cap \; Z \otimes Z \otimes I$$

$$\cap \; I \otimes I \otimes Z \; - \; Z_3$$

Measurement

Meas 1:

$$
\begin{aligned}
& X \otimes X \otimes I \\
\cap\ & Z \otimes Z \otimes I \\
\cap\ & I \otimes I \otimes Z
\end{aligned}
$$

Measurement

Meas 1:

$$Z_1$$

$$X \otimes X \otimes I$$

$$\cap \; Z \otimes Z \otimes I$$

$$\cap \; I \otimes I \otimes Z$$

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$$\cap \; X \otimes X \otimes I$$

$$\cap \; Z \otimes Z \otimes I$$

$$\cap \; I \otimes I \otimes Z$$

$$\equiv$$

$$Z_1 \cap Z_2 \cap Z_3$$

Adding T

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- On X, $TXT^\dagger = \frac{1}{\sqrt{2}}(X + Y)$, so $T : X \rightarrow \frac{1}{\sqrt{2}}(X + Y)$

Adding T

- For any U, $U : N \to UNU^\dagger$ (Gottesman 1998)
- For T, $TZT^\dagger = Z$ (trivially), so $T : Z \to Z$
- On X, $TXT^\dagger = \frac{1}{\sqrt{2}}(X + Y)$, so $T : X \to \frac{1}{\sqrt{2}}(X + Y)$
- We call these *additive types*.

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- Hence any unitary matrix can be given a type, once additive types are permitted.

- Conveniently, all the typing rules distribute over addition.

Typing $T^\dagger$

$T^\dagger = Z; S; T$

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$T^\dagger : Z \to Z$ (trivially)

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$T^\dagger = Z; S; T$

$T^\dagger : Z \to Z$ (trivially)

$Z : X \to -X$

Typing $T^\dagger$

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$T : -Y$

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$T^\dagger = Z; S; T$

$T^\dagger : Z \rightarrow Z$ (trivially)

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$S : -X \rightarrow -Y$

$T : -Y \ = -iXZ$

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$T : -Y = -iXZ \rightarrow -i\frac{1}{\sqrt{2}}(X+Y)Z$

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$T : -Y \; = -iXZ \; \to -i\frac{1}{\sqrt{2}}(X+Y)Z \; = \frac{1}{\sqrt{2}}(-Y+X)$

$$T^\dagger : (X \to \frac{1}{\sqrt{2}}(X - Y)) \cap (Z \to Z)$$

Typing Toffoli

```
TOFF := H 3; CNOT 2 3; T† 3; CNOT 1 3; T 3;
        CNOT 2 3; T† 3; CNOT 1 3; T 2; T 3;
        H 3; CNOT 1 2; T 1; T† 2; CNOT 1 2.
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- `TOFF` : $Z_1 \rightarrow Z_1 \cap Z_2 \rightarrow Z_2$, trivially.

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- `TOFF` : $Z_1 \rightarrow Z_1 \cap Z_2 \rightarrow Z_2$, trivially.
- `TOFF`: $Z_3 \rightarrow \frac{1}{2}(I \otimes I + Z \otimes I + I \otimes Z + Z \otimes Z) \otimes Z$

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- `TOFF` : $Z_1 \rightarrow Z_1 \cap Z_2 \rightarrow Z_2$, trivially.
- `TOFF`: $Z_3 \rightarrow \frac{1}{2}(I \otimes I + Z \otimes I + I \otimes Z + Z \otimes Z) \otimes Z$
- Hence `TOFF` : $Z_1 \cap Z_2 \cap Z_3 \rightarrow Z_1 \cap Z_2 \cap Z_3$

Gate Injection

Want: $U : (X \rightarrow aX + bY) \cap (Z \rightarrow Z)$

Have: $|m\rangle : aX + bY$

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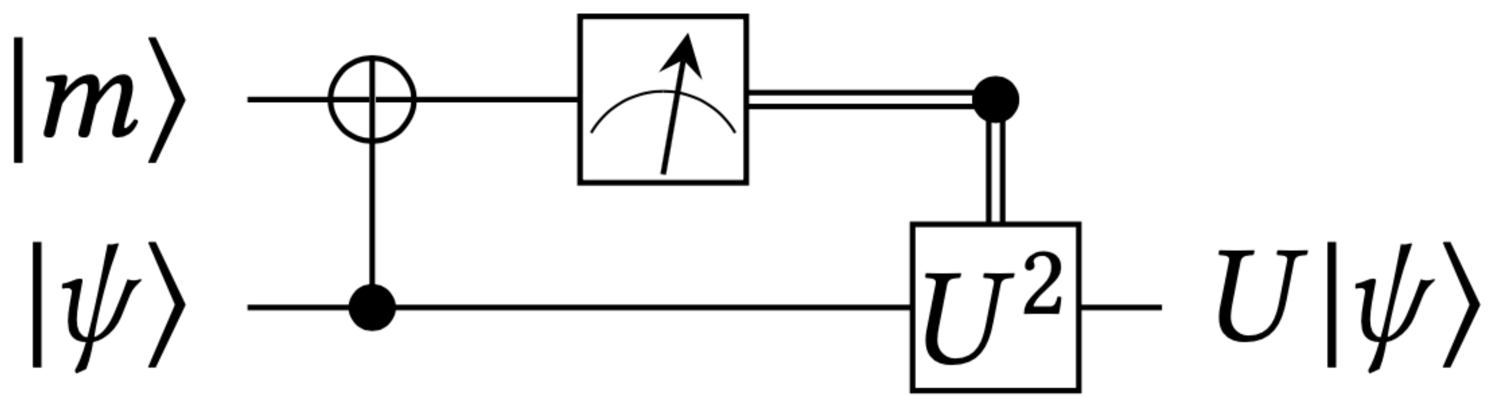

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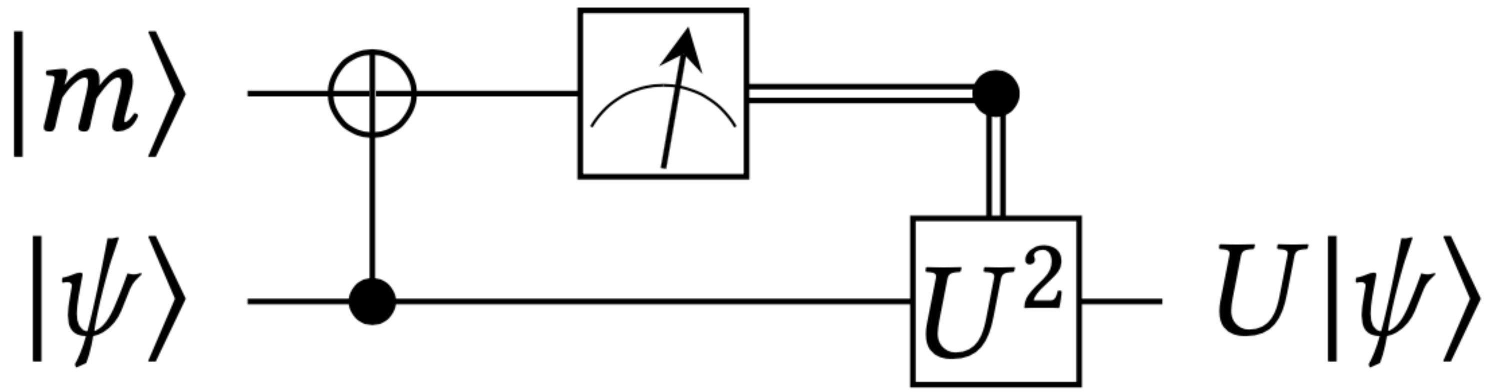

Get:

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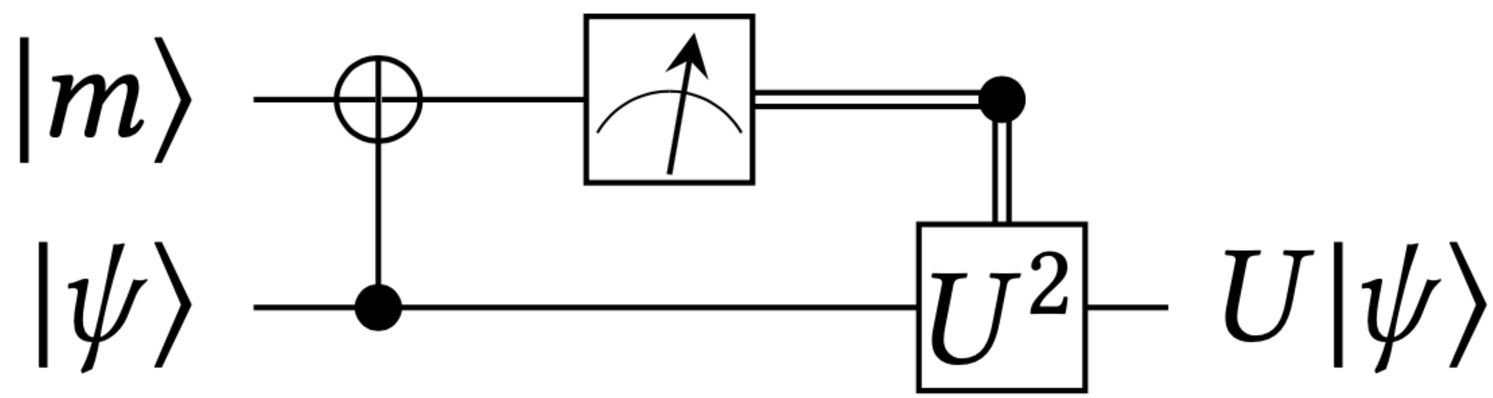

Get: $C : ((aX + bY)_1 \cap Z_2) \to Z$

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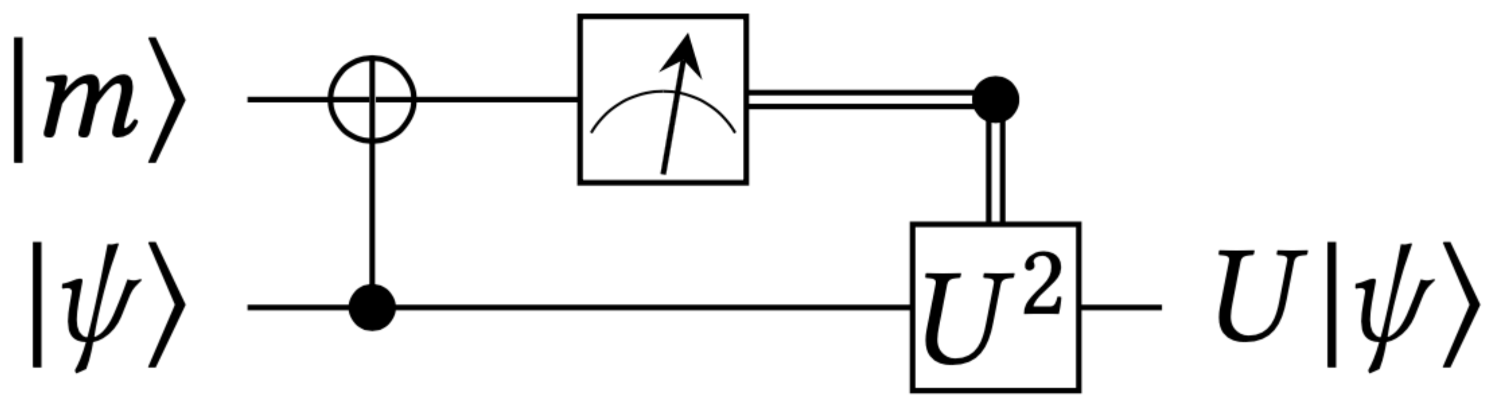
Have: $|m\rangle : aX + bY$



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- Consider *projections:* Eg. $\frac{1}{2}(I + Z)$ and $\frac{1}{2}(I - Z)$
- Types for error correcting codes.