Extending Gottesman Types Beyond the Clifford Group

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1 INTRODUCTION

In "Gottesman Types for Quantum Programs" [Rand et al. 2020], we proposed a type system for quantum circuits based on Gottesman's [1998] characterization of the Clifford set of gates H, S and $CNOT$. This type system interprets $H : \mathbf{Z} \rightarrow \mathbf{X}$ to say that the Hadamard gate H takes a qubit in the Z basis $\{|0\rangle, |1\rangle\}$ to a qubit in the X basis $\{|+\rangle, |-\rangle\}$, that is $X = HZH^\dagger$. We extended the system to allow us to take the intersection of types, which corresponds to conjunction. For instance, the Bell state $|\Phi^+\rangle$ has the type $(\mathbf{X} \otimes \mathbf{X}) \cap (\mathbf{Z} \otimes \mathbf{Z})$ (the joint eigenvectors of both components) and a Hadamard gate is fully specified by $H : (\mathbf{X} \rightarrow \mathbf{Z}) \cap (\mathbf{Z} \rightarrow \mathbf{X})$.

A key feature of this system and the extensions discussed below is that the base types correspond to matrices: in the original system, Pauli operators and in the extension, general Hermitian matrices. Notationally, we will use uppercase letters $U, V, \ldots$ for unitaries, and for a Hermitian matrix A, its corresponding type will be $\mathbf{A}$. It suffices to fully specify a gate's type by its actions on every permutation of $\mathbf{X}$ and $\mathbf{Z}$ inputs as they provide an information-theoretically complete description of the gate ($\mathbf{Y} = i\mathbf{X} * \mathbf{Z}$ because $Y = iXZ$). Figure 1 shows a representative set of typing rules for our original system.

Our type system also made judgments about the *separability* of a qubit from the rest of the quantum state. We used the notation $\mathbf{A} \times \mathbf{B}$ to represent a separable state, whose first component is in the A basis and whose second is in the B basis. With this, we gave a $CNOT$ gate the type $\mathbf{X} \times \mathbf{X} \rightarrow \mathbf{X} \times \mathbf{X}$ (among others), indicating that it preserves the separability of a pair of $\mathbf{X}$ qubits. In this work, we revise the separability judgments to allow for more precise reasoning about the separability across an arbitrary partition of a multi-qubit system, introduce typing judgments for measurement in the computational basis, along with a normalization technique that facilitates measurement, and extend the type system to a *universal* set of quantum gates, namely the Clifford+T group. In our prior work, we said that T has the type $\mathbf{Z} \rightarrow \mathbf{Z}$ but left its behavior, when applied to an $\mathbf{X}$ qubit, unspecified. Here we address this shortcoming by allowing *linear combination types* (or LC-types) like $\frac{1}{\sqrt{2}}(\mathbf{X} + \mathbf{Z})$. We apply our extended typing rules to fully derive the type of gate injection circuits [Nielsen and Chuang 2010, §10.6.2] for non-Clifford gates such as T.

A crucial difference from our previous work is a narrower interpretation of types along with their signs. Here, in contrast to the above, $+\mathbf{A}$ (resp. $-\mathbf{A}$) corresponds to the $+1$ (resp. -1) eigenspace of A. While this makes type inference for some terms more involved, it is essential to correctly type measurement for LC-types. It also finds application in the realm of quantum error-correcting codes, where the signs match the error syndromes.

The revised approach to separability and treatment of measurement, both on stabilizer states, appear in our January 2021 preprint [Rand et al. 2021], whereas the addition of T and LC-types is being presented for the first time here and is discussed in more detail.

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$$H : (\mathbf{X} \rightarrow \mathbf{Z}) \cap (\mathbf{Z} \rightarrow \mathbf{X}) \qquad S : (\mathbf{X} \rightarrow \mathbf{Y}) \cap (\mathbf{Z} \rightarrow \mathbf{Z})$$
$$CNOT : (\mathbf{X} \otimes \mathbf{I} \rightarrow \mathbf{X} \otimes \mathbf{X}) \cap (\mathbf{I} \otimes \mathbf{X} \rightarrow \mathbf{I} \otimes \mathbf{X}) \cap (\mathbf{Z} \otimes \mathbf{I} \rightarrow \mathbf{Z} \otimes \mathbf{I}) \cap (\mathbf{I} \otimes \mathbf{Z} \rightarrow \mathbf{Z} \otimes \mathbf{Z})$$
$$\frac{g : \mathbf{A} \qquad g : \mathbf{B}}{g : \mathbf{A} \cap \mathbf{B}}\ \cap\text{-I} \qquad \frac{g : \mathbf{A} \cap \mathbf{B}}{g : \mathbf{A}}\ \cap\text{-E}$$
$$\frac{g : \mathbf{A} \rightarrow \mathbf{A}' \qquad g : \mathbf{B} \rightarrow \mathbf{B}'}{g : \mathbf{A} * \mathbf{B} \rightarrow \mathbf{A}' * \mathbf{B}'}\ * \qquad \frac{g : \mathbf{A} \rightarrow \mathbf{B}}{g : \mathbf{A} \otimes \mathbf{I} \rightarrow \mathbf{B} \otimes \mathbf{I}}\ \otimes$$
$$\frac{p_1 : \mathbf{A} \rightarrow \mathbf{B} \qquad p_2 : \mathbf{B} \rightarrow \mathbf{C}}{p_1; p_2 : \mathbf{A} \rightarrow \mathbf{C}}\ \text{CUT} \qquad \frac{p : \mathbf{A} \rightarrow \mathbf{A}' \qquad c \in \{-1, i\}}{p : c\mathbf{A} \rightarrow c\mathbf{A}'}\ \text{SCALE}$$

Fig. 1. Some sample types and typing rules

2 SEPARABILITY FOR STABILIZER STATES

In our previous work, we showed that the type $\mathbf{I}^m \otimes \mathbf{A} \otimes \mathbf{I}^n$ is separable into the $m+1^{\text{th}}$ qubit and the remainder of the state [Rand et al. 2020, Corollary 1]. Here we introduce the notation $\mathbf{A}_i$ (for any type $\mathbf{A}$) to represent a state where the i^{th} qubit is separable from the rest of the system and is an eigenvector of A. For example, the two qubit system $|0\rangle \otimes |+\rangle$ has the type $\mathbf{Z}_1 \cap \mathbf{X}_2$.

We explain how to generalize beyond single qubit separability with the following example. Consider a 2-qubit type $(\mathbf{X} \otimes \mathbf{X} \cap \mathbf{Z} \otimes \mathbf{Z})$ whose joint eigenspace is spanned by the two maximally entangled Bell states $\{|\Phi^+\rangle, |\Psi^-\rangle\}$. Also, consider an n-qubit state with this type on the first and third qubits. Being maximally entangled, these qubits should be disjoint from the rest of the system and hence, its type is $(\mathbf{X} \otimes \mathbf{X} \cap \mathbf{Z} \otimes \mathbf{Z})_{1,3}$. If the second and fourth qubits are similarly entangled, the system has type $(\mathbf{X} \otimes \mathbf{X} \cap \mathbf{Z} \otimes \mathbf{Z})_{1,3} \cap (\mathbf{X} \otimes \mathbf{X} \cap \mathbf{Z} \otimes \mathbf{Z})_{2,4}$. In general, to separate k out of n qubits, we require that the intersection type has a joint eigenspace of dimension 2^{k-1} on the k qubits. This is satisfied if there exist k pairwise commuting terms where each term is the tensor products of Paulis on the k qubits along with $\mathbf{I}$'s on the rest of the system [Rand et al. 2021, Proposition 4.3].

3 TYPING MEASUREMENT FOR STABILIZER STATES

To understand the intuition behind typing measurements, it helps to identify intersection types with the joint eigenspaces of their matrices. Then, unitary evolution preserves this structure by rotating each eigenspace equally, which can be deduced term by term. However, the application of a measurement operation could completely change the structure of the joint eigenspace. Hence, finding the post-measurement type may require modifications to the type as a whole.

For simplicity, let us consider measuring the first qubit of an n-qubit stabilizer state in the Z-basis with outcome +1.[1] First, observe that the measurement collapses the state to be an eigenstate of $(+\mathbf{Z})_1$ and so this type should be added as a term to the intersection. As the resulting state is separable in the first factor, it takes the form $|0\rangle \otimes |\psi\rangle$.

Any term $\mathbf{A}$ in the intersection type that commutes with $(+\mathbf{Z})_1$ already has joint eigenvectors with it and so can be left unchanged, except perhaps for its sign which would need to be updated to be the eigenvalue of $|0\rangle \otimes |\psi\rangle$.

However, a term in the intersection that anti-commutes with $(+\mathbf{Z})_1$ must have a leading factor of $\mathbf{X}$ or $\mathbf{Y}$. Such a term cannot have $|0\rangle \otimes |\psi\rangle$ as an eigenvector as it will not fix the leading state to $|0\rangle$. Thus it must be removed from the intersection, or equivalently replaced by $\mathbf{I}^n$.

[1] The analysis is similar for a -1 outcome with appropriate sign changes.

The critical idea, as laid out in Gottesman [1998, §7], is that if there are multiple anti-commuting terms in the intersection, then we do not remove them all. Rather, we note that if two terms both anti-commute with $\mathbf{Z}_1$ then their product will commute with $\mathbf{Z}_1$. Thus, we can rewrite the intersection type using the rule $\mathbf{A} \cap \mathbf{B} = \mathbf{A} \cap \mathbf{A} * \mathbf{B}$.[2] Moreover, in the case of stabilizer states, it is known that there will be at most one anti-commuting term that will have to be removed, making the calculation of the post-measurement state efficient.

4 TYPES FOR UNIVERSAL QUANTUM COMPUTATION

To type non-Clifford gates, we need to expand our type system to include LC-types, i.e., linear combinations of $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$. For any operator C, and Pauli operators P, Q, we have $C(P+Q)C^\dagger = CPC^\dagger + CQC^\dagger$, and so we can derive arrow types beyond stabilizer states associated to Pauli operators. Consider the set $\mathcal{M}$ of one-qubit Hermitian operators with eigenvalues +1 and −1. Clearly $X, Y, Z \in \mathcal{M}$, and in fact,

$$\mathcal{M} = \{aX + bY + cZ \;:\; a, b, c \in \mathbb{R} \text{ with } a^2 + b^2 + c^2 = 1\}.$$

This is just another presentation of the Bloch sphere. More directly, any one-qubit unitary has the form [Nielsen and Chuang 2010, Equation 4.8]:

$$U = tI + iaX + ibY + icZ$$

where $t, a, b, c \in \mathbb{R}$ with $t^2 + a^2 + b^2 + c^2 = 1$. Taking $t \to 0$ gives $U = iaX + ibY + icZ$, which is clearly also anti-Hermitian ($U^\dagger = -U$). But the leading i is just a global phase, which we may drop to obtain an element of $\mathcal{M}$.

For any $M = aX + bY + cZ \in \mathcal{M}$ we associate a type $\mathbf{M}$, consisting of the eigenstate $|m\rangle$ (with type $-\mathbf{M}$ consisting of the (-1)-eigenstate $|-m\rangle$). Any 1-qubit Clifford operator has the arrow type

$$C : \mathbf{M} \to aC(\mathbf{X}) + ibC(\mathbf{X}) * C(\mathbf{Z}) + cC(\mathbf{Z}).$$

The above arguments hold for any unitary, not just Clifford operators. So we can extend Gottesman types to non-Clifford operators by allowing the codomain of the arrow to be types associated to operators in $\mathcal{M}$. For example, the T gate, which in Rand et al. [2020] had type $\mathbf{X} \to \top$, now has the full intersection type (see, for example, Matsumoto and Amano [2008, Equation 12])

$$T : (\mathbf{X} \to \tfrac{1}{\sqrt{2}}(\mathbf{X} + \mathbf{Y})) \cap (\mathbf{Z} \to \mathbf{Z}).$$

We can easily show $T : \mathbf{Y} \to \frac{1}{\sqrt{2}}(\mathbf{Y} - \mathbf{X})$, but we can also derive other arrow types for T on non-stabilizer states. Using our $\mathbf{M}$ from above,

$$T : \mathbf{M} = a\mathbf{X} + b\mathbf{Y} + c\mathbf{Z} \to \tfrac{a}{\sqrt{2}}(\mathbf{X} + \mathbf{Y}) + \tfrac{b}{\sqrt{2}}(\mathbf{Y} - \mathbf{X}) + c\mathbf{Z}.$$

For instance, we can now derive the type for $S = T; T$ on $\mathbf{X}$ as

$$\mathbf{X} \to \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(\mathbf{X} + \mathbf{Y}) + \frac{1}{\sqrt{2}}(\mathbf{Y} - \mathbf{X})\right) = \frac{1}{2}\mathbf{Y} + \frac{1}{2}\mathbf{Y} = \mathbf{Y}$$

and trivially $T; T : \mathbf{Z} \to \mathbf{Z}$. We can similarly derive that $T^\dagger$ (a sequence of seven T's or $Z; S; T$) has type $\mathbf{X} \to \frac{1}{\sqrt{2}}(\mathbf{X} - \mathbf{Y})$.

Measurement of general LC-types can be complicated. However for certain instances, we can provide concrete rules for how types transform under measurement. Consider measuring the first qubit in the Z basis using a state of type

$$|0\rangle \otimes |\psi\rangle + |1\rangle \otimes |\phi\rangle \;:\; a\mathbf{I} \otimes \mathbf{K} + b\mathbf{Z} \otimes \mathbf{L}.$$

[2] This holds as the joint eigenspace of A and B is the same as that of A and AB.

This means that the given state is a (+1)-eigenstate of the associated operator:

$$(aI \otimes K + bZ \otimes L)(|0\rangle \otimes |\psi\rangle + |1\rangle \otimes |\phi\rangle) = |0\rangle \otimes |\psi\rangle + |1\rangle \otimes |\phi\rangle\,.$$

But a direct computation gives

$$(aI \otimes K + bZ \otimes L)(|0\rangle \otimes |\psi\rangle + |1\rangle \otimes |\phi\rangle) = |0\rangle \otimes (aK + bL)\,|\psi\rangle + |1\rangle \otimes (aK - bL)\,|\phi\rangle\,.$$

Comparing terms: $(aK+bL)\,|\psi\rangle = |\psi\rangle$ and $(aK-bL)\,|\phi\rangle = |\phi\rangle$. Therefore we may claim that the post-measurement type leaves this term unchanged. With measurement projections $\Pi_0^Z = \frac{1}{2}(I^n + Z \otimes I^{n-1})$ and $\Pi_1^Z = \frac{1}{2}(I^n - Z \otimes I^{n-1})$, regardless of the observed outcome

$$\Pi_0^Z(|0\rangle \otimes |\psi\rangle + |1\rangle \otimes |\phi\rangle) = |0\rangle \otimes |\psi\rangle : a\mathbf{I} \otimes \mathbf{K} + b\mathbf{Z} \otimes \mathbf{L}, \text{ or}$$
$$\Pi_1^Z(|0\rangle \otimes |\psi\rangle + |1\rangle \otimes |\phi\rangle) = |1\rangle \otimes |\phi\rangle : a\mathbf{I} \otimes \mathbf{K} + b\mathbf{Z} \otimes \mathbf{L}.$$

Note that we purposefully used $\mathbf{Z}$ to fix the sign differences between the measurement outcomes.[3]

Yet, by correcting the sign in the measurement outcome, we have obfuscated a critical feature: after measurement, the measured qubit is separable from the rest of the system. Revisiting the above computation

$$\Pi_0^Z(|0\rangle \otimes |\psi\rangle + |1\rangle \otimes |\phi\rangle) = |0\rangle \otimes |\psi\rangle : a\mathbf{I} \otimes \mathbf{K} + b\mathbf{I} \otimes \mathbf{L} = \mathbf{I} \otimes (a\mathbf{K} + b\mathbf{L})$$
$$\Pi_1^Z(|0\rangle \otimes |\psi\rangle + |1\rangle \otimes |\phi\rangle) = |1\rangle \otimes |\phi\rangle : a\mathbf{I} \otimes \mathbf{K} - b\mathbf{I} \otimes \mathbf{L} = \mathbf{I} \otimes (a\mathbf{K} - b\mathbf{L}).$$

Since we additionally know the state is of type $\pm\mathbf{Z}_1$, where the sign is the measurement outcome, we can conclude that the post-measurement state has type

$$\begin{cases} \mathbf{Z}_1 \cap (a\mathbf{K} + b\mathbf{L})_{2,\ldots,n} & \text{if 0 was measured,} \\ -\mathbf{Z}_1 \cap (a\mathbf{K} - b\mathbf{L})_{2,\ldots,n} & \text{if 1 was measured.} \end{cases}$$

That is, we have identified the post-measurement type (conditioned on the outcome of the measurement) of an $(n-1)$-qubit factor in our n-qubit system. While this may be troublesome from the perspective of static typechecking, many quantum algorithms and protocols rely on post-measurement corrections, as we will illustrate in the next section.

5 EXAMPLE: GATE INJECTION THROUGH MAGIC STATES

A standard approach to universal quantum computation implements non-Clifford gates on quantum codes using "magic" states through a method called gate injection. For concreteness, we focus on 1-qubit rotations about the Z-axis, by which we mean unitaries of type

$$U : (\mathbf{X} \to \mathbf{M}) \cap (\mathbf{Z} \to \mathbf{Z}) \tag{1}$$

for some $M \in \mathcal{M}$. While we could write explicit matrix identities for such unitaries, let us see what we can derive by simply appealing to typing judgements. Note that these arrow types mean $UZU^\dagger = Z$ and $UXU^\dagger = M$. Since X and Z anti-commute, so do M and Z and therefore $M = aX + bY$ where $a^2 + b^2 = 1$. Unsurprisingly, T fits this mold with $a = b = \frac{1}{\sqrt{2}}$. We will parameterize $a = \cos\theta$ and $b = \sin\theta$; by direct computation $U : \mathbf{Y} \to -\sin\theta \cdot \mathbf{X} + \cos\theta \cdot \mathbf{Y}$, so U acts to rotate types in the X/Y-plane of the Bloch sphere by an angle θ.

We claim that we can implement U using the state $|m\rangle : \mathbf{M}$ with the circuit of Figure 2. Note that U has the type from Equation (1), but the circuit of Figure 2 is a 2-qubit gate and so we would like to show it has type $(\mathbf{Z}_2 \to \mathbf{Z}_2) \cap (\mathbf{X}_2 \to \mathbf{M}_2)$. Specifically, we prove the circuit is of type

$$(\mathbf{M}_1 \cap \mathbf{Z}_2 \to \pm\mathbf{Z}_1 \cap \mathbf{Z}_2) \cap (\mathbf{M}_1 \cap \mathbf{X}_2 \to \pm\mathbf{Z}_1 \cap \mathbf{M}_2)$$

where the sign in this type is the outcome of the measurement.

[3]Namely, $|1\rangle \otimes |\phi\rangle : a\mathbf{I} \otimes \mathbf{K} - b(-\mathbf{Z}) \otimes \mathbf{L} = a\mathbf{I} \otimes \mathbf{K} + b\mathbf{Z} \otimes \mathbf{L}$.

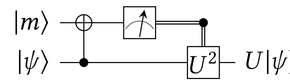

Fig. 2. Gate injection circuit for Z-type rotations U.

Beginning with $\mathbf{M}_1 \cap \mathbf{Z}_2$ which is equivalent to $\mathbf{M} \otimes \mathbf{I} \cap \mathbf{I} \otimes \mathbf{Z}$ we evaluate the effect of the circuit on each term of the intersection:

$$(\cos\theta \cdot \mathbf{X} + \sin\theta \cdot \mathbf{Y}) \otimes \mathbf{I} \xrightarrow{NOTC} \cos\theta \cdot \mathbf{X} \otimes \mathbf{I} + \sin\theta \cdot \mathbf{Y} \otimes \mathbf{Z} \xrightarrow{Meas_1} \mathbf{I}^2$$
$$\mathbf{I} \otimes \mathbf{Z} \xrightarrow{NOTC} \mathbf{I} \otimes \mathbf{Z} \xrightarrow{Meas_1} \mathbf{I} \otimes \mathbf{Z}.$$

And so depending on the sign of our measured first qubit we get the post-measurement type

$$\mathbf{I}^2 \cap (\mathbf{I} \otimes \mathbf{Z}) \cap (\pm\mathbf{Z} \otimes \mathbf{I}) = \pm\mathbf{Z}_1 \cap \mathbf{Z}_2.$$

Now turning to the case $\mathbf{M}_1 \cap \mathbf{X}_2 = \mathbf{M} \otimes \mathbf{I} \cap \mathbf{I} \otimes \mathbf{X}$ we evaluate:

$$(\cos\theta \cdot \mathbf{X} + \sin\theta \cdot \mathbf{Y}) \otimes \mathbf{I} \xrightarrow{NOTC} \cos\theta \cdot \mathbf{X} \otimes \mathbf{I} + \sin\theta \cdot \mathbf{Y} \otimes \mathbf{Z}$$
$$\mathbf{I} \otimes \mathbf{X} \xrightarrow{NOTC} \mathbf{X} \otimes \mathbf{X}.$$

Now, however, our input to the measurement

$$(\cos\theta \cdot \mathbf{X} \otimes \mathbf{I} + \sin\theta \cdot \mathbf{Y} \otimes \mathbf{Z}) \cap (\mathbf{X} \otimes \mathbf{X})$$

has too many terms with an $\mathbf{X}$ in the first factor. Multiplying the second term into the first has

$$(\cos\theta \cdot \mathbf{I} \otimes \mathbf{X} + \sin\theta \cdot \mathbf{Z} \otimes \mathbf{Y}) \cap (\mathbf{X} \otimes \mathbf{X}).$$

Now we apply the conclusion of the previous section to write the post-measurement state as

$$\begin{cases} \mathbf{Z}_1 \cap (\cos\theta \cdot \mathbf{X} + \sin\theta \cdot \mathbf{Y})_2 & \text{if 0 was measured,} \\ -\mathbf{Z}_1 \cap (\cos\theta \cdot \mathbf{X} - \sin\theta \cdot \mathbf{Y})_2 & \text{if 1 was measured.} \end{cases}$$

So we see that upon measuring 0 the resulting state is of type $\mathbf{Z}_1 \cap \mathbf{M}_2$ as desired. But upon measuring 1 we have resulting type $-\mathbf{Z}_1 \cap (\cos(-\theta) \cdot \mathbf{X} + \sin(-\theta) \cdot \mathbf{Y})_2$, and so have accomplished the rotation in the opposite direction. That is, we have implemented $U^\dagger$ and so doing a post-selected correction of U^2, as shown in Figure 2, produces output type $-\mathbf{Z}_1 \cap \mathbf{M}_2$ as desired.

We have shown that the circuit of Figure 2 implements the operator U without ever applying that operator, though U^2 is applied as a post-measurement correction. For example, setting U to T gives us a non-Clifford unitary leading to universal computation while $U^2 = S$ is Clifford.

6 TYPECHECKING EFFICIENCY

Finally, we address the question of how efficiently quantum circuits can be typechecked. Stabilizer states evolving under Clifford operations are known to have succinct canonical representations [Aaronson and Gottesman 2004]. Similarly, pure unitary evolution in our system can be linearly typechecked and our row echelonization-inspired normalization procedure [Rand et al. 2021, §5.1] allows us to typecheck circuits with measurements in $O(GN + N^2)$ time, where N is the number of qubits and G the number of gates. By contrast, our LC-types cannot be efficiently typechecked in the general case, unless $P = BQP$. That said, certain common Clifford+T circuits, like the Toffoli gate applied to classical qubits, can be efficiently typechecked as the LC-types simplify to Paulis. We leave characterizing the set of circuits that our expanded type system can check efficiently for future work.

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