VPHL: A Verified Partial-Correctness Logic for Probabilistic Programs

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Mathematical Foundations of Programming Semantics XXXI

VPHL

Verified
Probabilistic
Hoare
Logic

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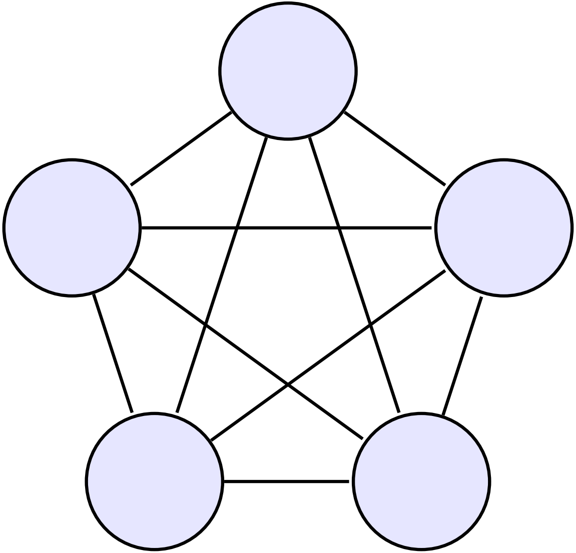
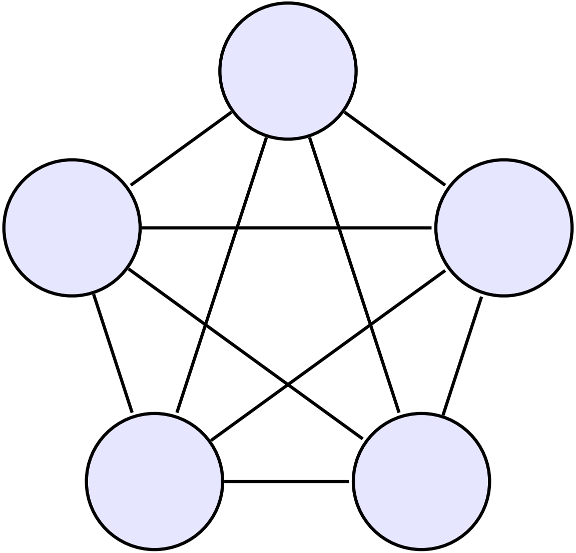
Let's Take a Random Walk...



Rabbit Hunting



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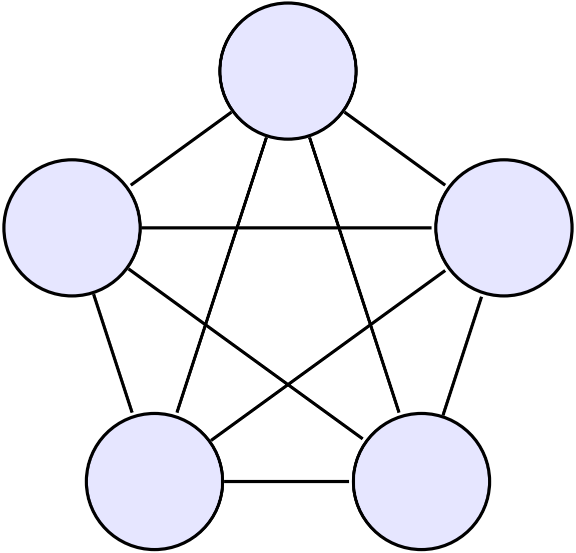

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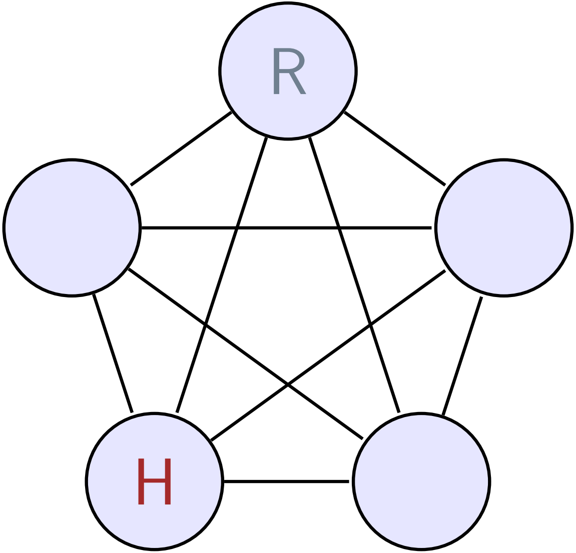

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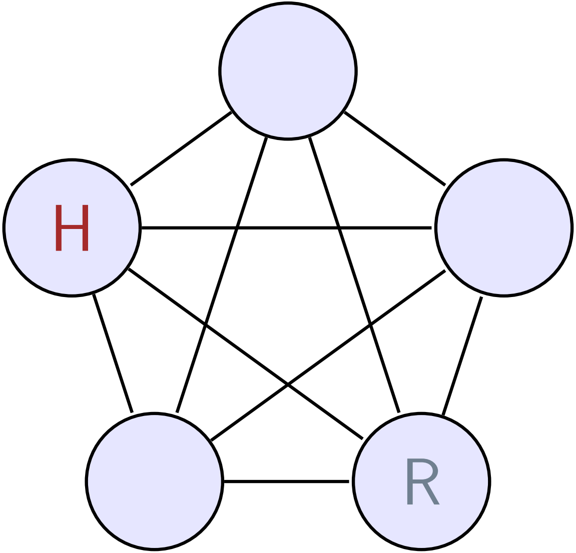

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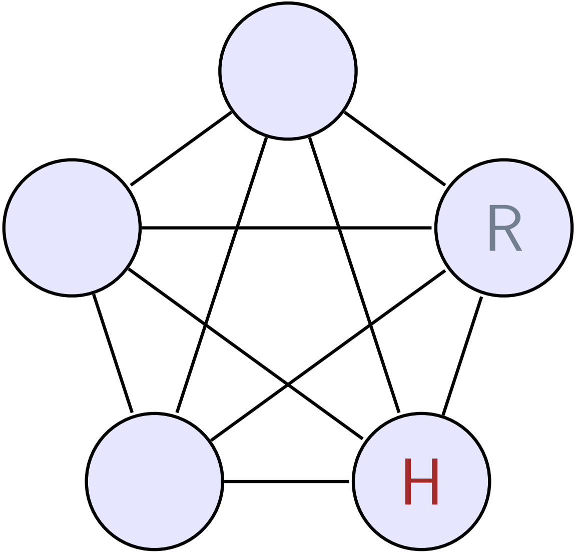

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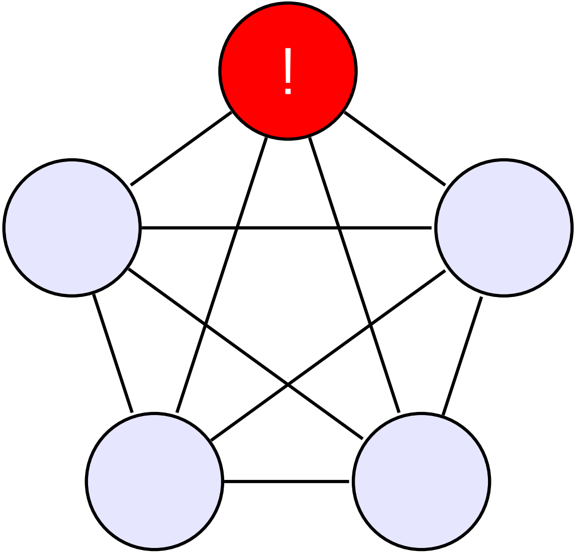


A Program to Analyze

Rabbit Hunting

i := 0
caught := F
while $i < n$ **do**
    rabbit := UNIFORM(k)
    hunter := UNIFORM(k)
    caught := $caught \vee (hunter = rabbit)$
    i := $i + 1$
end while

A Program to Analyze

Rabbit Hunting

$\{Pr(True) = 1\}$
$i := 0$
$caught := \texttt{F}$
while $i < n$ **do**
  $rabbit := \texttt{UNIFORM}(k)$
  $hunter := \texttt{UNIFORM}(k)$
  $caught := caught \vee (hunter = rabbit)$
  $i := i + 1$
end while
$\{Pr(caught) = ?\}$

Comparison

Paper	Full Distributions	While Loops
Ramshaw, 1979	No	Partial
Den Hartog & De Vink, 2002	No	Partial
Chadha et. al., 2007	No	No
VPHL	Yes	Partial

Principles

- Simple
  - Full Distributions
  - Truth-functional propositions
  - Resembles standard Hoare-logic
- Reliable
  - Rigorously verified deductive system
  - Can be safely extended
- Powerful
  - Support for non-termination
  - Capable of analyzing standard randomized algorithms

A Probabilistic Language

Classic Imperative Language *Imp*:

$$\theta : id \rightarrow value$$

Probabilistic Imperative Language *PrImp*:

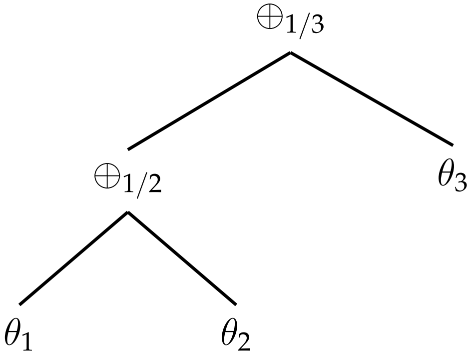
$$\Theta : \theta \rightarrow [0, 1]$$

Full Distributions with Finite Support

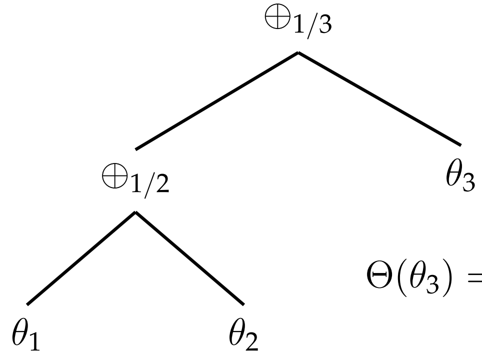
$$\sum_{\theta} \Theta(\theta) = 1$$

Requiring finite support it allows us to represent distributions using a simple inductive structure.

Representing Distributions



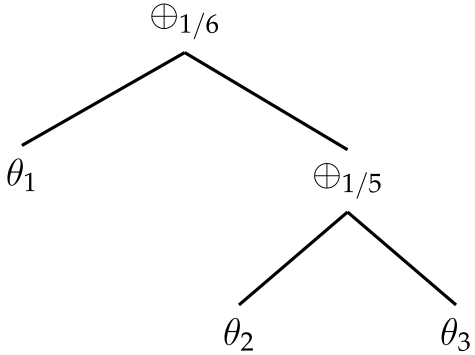
Representing Distributions



$\Theta(\theta_3) = \frac{2}{3}$

$\Theta(\theta_1) = \frac{1}{6}$ $\quad$ $\Theta(\theta_2) = \frac{1}{6}$

Representing Distributions



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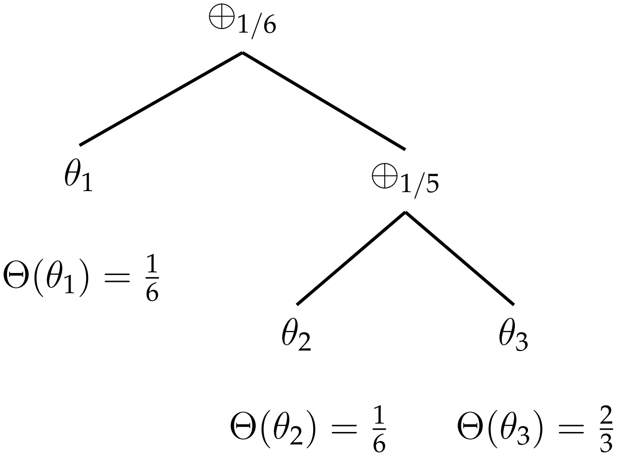

Probability

For a boolean expression b and distribution Θ:

$$Pr_{\Theta}(b) = \sum_{\theta} \{\Theta(\theta) \mid b \text{ is true in } \theta\}$$

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$$\Theta(\theta_1) = 1/6 \qquad \Theta(\theta_2) = 1/6 \qquad \Theta(\theta_3) = 2/3$$

$$\theta_1(x) = 1 \qquad \theta_2(x) = 2 \qquad \theta_3(x) = 3$$

Probability

For a boolean expression b and distribution Θ:

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$\Theta(\theta_1) = 1/6$ $\Theta(\theta_2) = 1/6$ $\Theta(\theta_3) = 2/3$

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$$Pr_\Theta(x \text{ odd})$$

Probability

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$\theta_1(x) = 1$ $\quad$ $\theta_2(x) = 2$ $\quad$ $\theta_3(x) = 3$

$$Pr_{\Theta}(x \text{ odd}) = 1/6 + 2/3 = 5/6$$

Tautology

For any distribution Θ and tautology T:

$$Pr_{\Theta}(T) = 1$$

Complement

For any distribution Θ and boolean b:

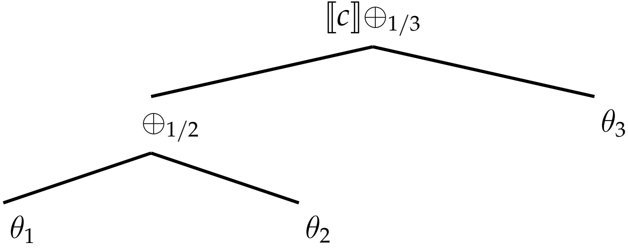
$$Pr_\Theta(\neg b) = 1 - Pr_\Theta(b)$$

Marginalization

For any distribution Θ and booleans a, b:

$$Pr_{\Theta}(a) = Pr_{\Theta}(a \wedge b) + Pr_{\Theta}(a \wedge \neg b)$$

Primp Commands



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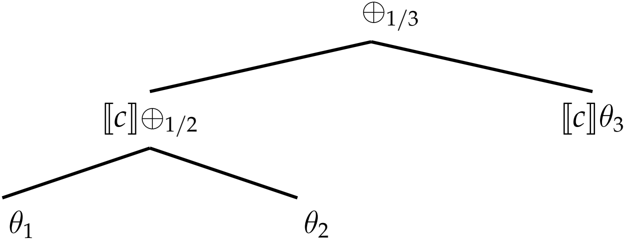

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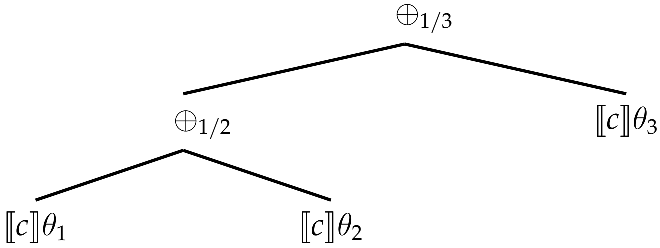
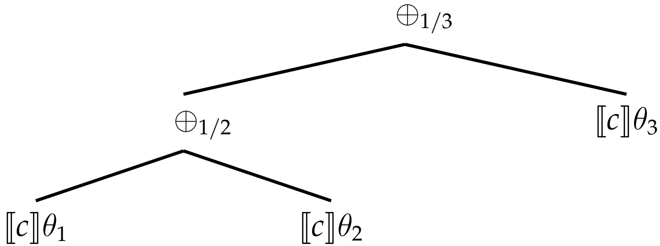


Primp Commands

$c \equiv y$ `:=` $\texttt{toss}(\frac{1}{5})$



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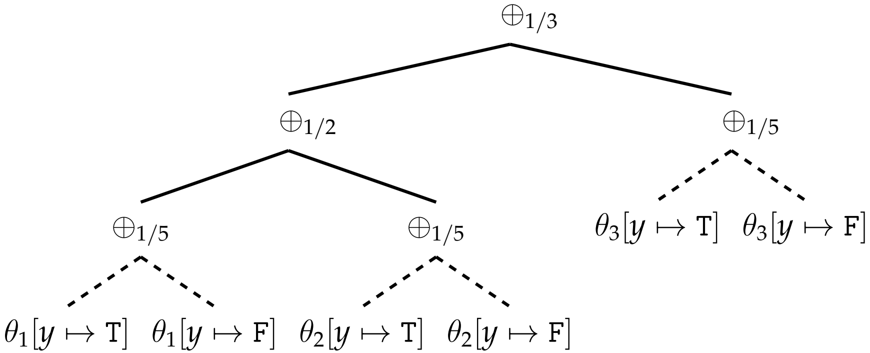

VPHL: Hoare Logic

Definition: $\{P\}\ c\ \{Q\}$

$$\frac{P(\Theta) \qquad c \,/\, \Theta \Downarrow \Theta'}{Q(\Theta')}$$

Truth-functional assertions over full distributions

$$\mathcal{P}, \mathcal{Q} ::= Pr(\mathcal{B}) = p \mid Pr(\mathcal{B}) < p \mid Pr(\mathcal{B}) > p \\ \mid \mathcal{P} \wedge \mathcal{P} \mid \mathcal{P} \vee \mathcal{P}$$

Basic Rules

$$\frac{P' \to P \qquad \{P\}\ c\ \{Q\} \qquad Q \to Q'}{\{P'\}\ c\ \{Q'\}}\ \text{Consequence}$$

$$\text{Skip}\ \frac{}{\{P\}\ \texttt{skip}\ \{P\}} \qquad \frac{}{\{P[z \mapsto e]\}\ z\ \texttt{:=}\ e\ \{P\}}\ \text{Assign}$$

$$\frac{\{P\}\ c_1\ \{Q\} \qquad \{Q\}\ c_2\ \{R\}}{\{P\}\ c_1;\ c_2\ \{R\}}\ \text{Sequence}$$

The Toss Rule

$$\frac{y \text{ free in } P}{\{P\}\ y\ \texttt{:=}\ \texttt{toss}(p)\ \{P \triangleleft_p^y\}}\ \text{Toss}$$

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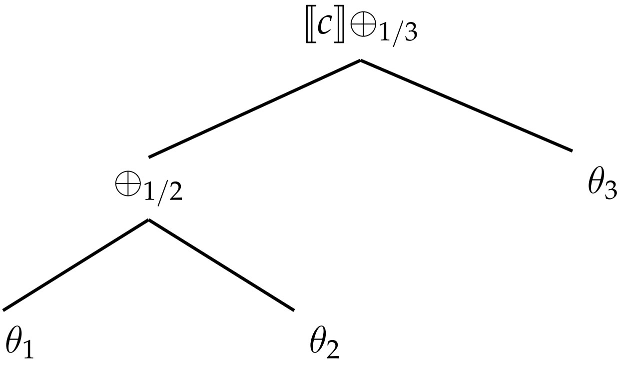
$$\frac{y \text{ free in } P}{\{P\}\ y\ \texttt{:=}\ \texttt{toss}(p)\ \{P \lhd_p^y\}}\ \text{Toss}$$

$$[Pr(b) = a]\lhd_p^y \equiv \qquad \begin{array}{l} Pr(b \wedge y) = pa \quad \wedge \\ Pr(b \wedge \neg y) = (1-p)a \end{array}$$

The If Rule

$c \equiv \texttt{if}\ y\ \texttt{then}\ c_1\ \texttt{else}\ c_2$



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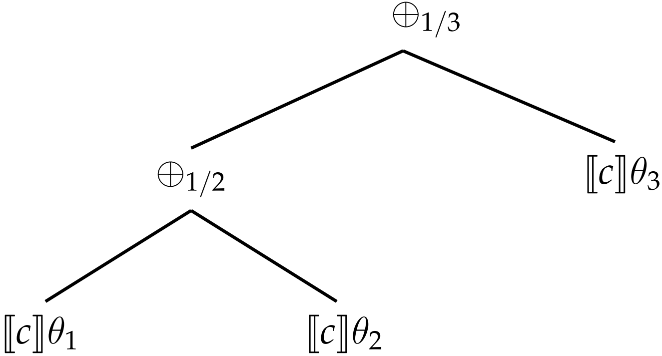

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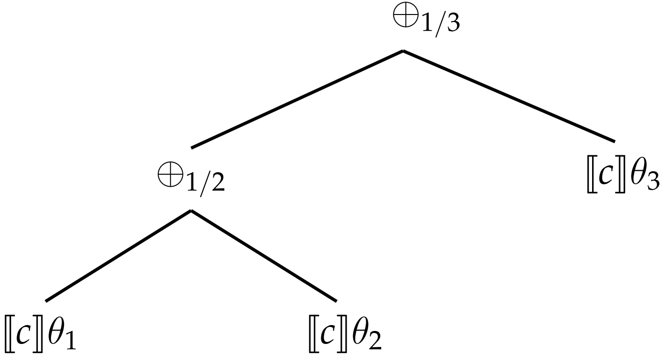

where $\theta_1(y) = \texttt{T}$, $\theta_2(y) = \texttt{F}$ and $\theta_3(y) = \texttt{F}$

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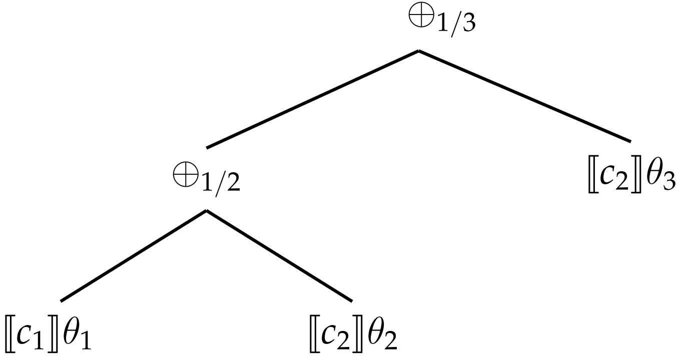

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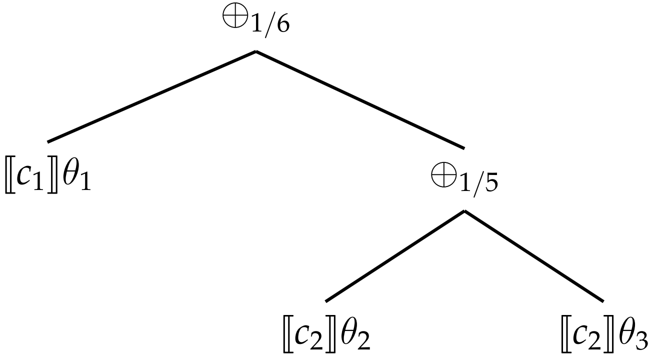

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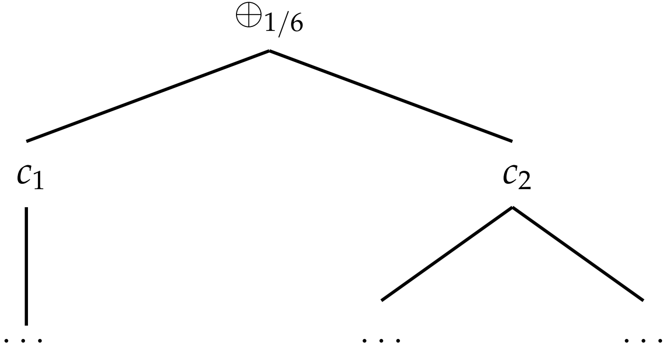

The If Rule

$c \equiv \texttt{if}\, y\, \texttt{then}\, c_1\, \texttt{else}\, c_2$

$$\oplus_{1/6}$$

$\{P_1 \wedge y\}\ c_1\ \{Q_1\}$ $\qquad$ $\{P_2 \wedge \neg y\}\ c_2\ \{Q_2\}$

$\dots$ $\qquad$ $\dots$ $\qquad$ $\dots$

The If Rule

$$c \equiv \texttt{if}\, y\, \texttt{then}\, c_1\, \texttt{else}\, c_2$$

$$\{Pr(y) = p \wedge P_1' \wedge P_2'\}\ c\ \{Q_1' \wedge Q_2'\}$$

$$\{P_1 \wedge y\}\ c_1\ \{Q_1\} \qquad\qquad \{P_2 \wedge \neg y\}\ c_2\ \{Q_2\}$$

$$\ldots \qquad\qquad \ldots \qquad\qquad \ldots$$

The If Rule

Why P_1'?

- Scaling – we have to normalize the probabilities in each branch

- Conditioning on the guard – we need to avoid conflict

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- Scaling – we have to normalize the probabilities in each branch
  $$Pr(b) = a \Rightarrow Pr(b) = p * a$$
- Conditioning on the guard – we need to avoid conflict
  $$Pr(b) = p * a \Rightarrow Pr(b \wedge y) = p * a$$

Applying the IF Rule

UNIFORM(3)

u_1 := $\texttt{toss}(\frac{1}{3})$;
if u_1 **then**
  x := 3
else
  u_2 := $\texttt{toss}(\frac{1}{2})$;
  if u_2 **then**
    x := 2
  else
    x := 1
  end if

end if

Applying the IF Rule

UNIFORM(3)

u_1 `:=` $\texttt{toss}(\frac{1}{3})$;
if u_1 **then**
    $\{Pr(3 = 3) = 1\}\ x \mathrel{:=} 3\ \{Pr(x = 3) = 1\}$
else
    u_2 `:=` $\texttt{toss}(\frac{1}{2})$;
    if u_2 **then**
        $\{Pr(2 = 2) = 1\}\ x \mathrel{:=} 2\ \{Pr(x = 2) = 1\}$
    else
        $\{Pr(1 = 1) = 1\}\ x \mathrel{:=} 1\ \{Pr(x = 1) = 1\}$
    end if

end if

Applying the IF Rule

UNIFORM(3)

$\{Pr(True) = 1\}\ u_1$ `:=` `toss`$(\frac{1}{3})$; $\{Pr(True \wedge u_1) = \frac{1}{3}\}$
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  $\{Pr(True) = 1\}\ u_2$ `:=` `toss`$(\frac{1}{2})$; $\{Pr(True \wedge u_2) = \frac{1}{2}\}$
  if u_2 **then**
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  else
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  end if

end if

Applying the IF Rule

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    end if

end if

Applying the IF Rule

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Applying the IF Rule

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  end if
  $\{Pr(x = 2 \wedge u_2) = \frac{1}{2} \wedge Pr(x = 1 \wedge \neg u_2) = \frac{1}{2}\}$
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  end if
  $\{Pr(x = 2) \geq \frac{1}{2} \wedge Pr(x = 1) \geq \frac{1}{2}\}$
end if

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$\quad \{Pr(True) = 1\}\ u_2 := \texttt{toss}(\frac{1}{2});\ \{Pr(u_2) = \frac{1}{2}\}$

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$\quad$ **end if**

$\quad \{Pr(x = 2) = \frac{1}{2} \wedge Pr(x = 1) = \frac{1}{2}\}$

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    end if
    $\{Pr(x = 2) = \frac{1}{2} \wedge Pr(x = 1) = \frac{1}{2}\}$
end if
$\{Pr(x = 3) = \frac{1}{3} \wedge Pr(x = 2) = \frac{1}{3} \wedge Pr(x = 1) = \frac{1}{3}\}$

The While Rule

We want to guarantee that the program terminates in some number of steps n, assuming that it terminates.

The *Deterministic Invariant* guarantees that the guard takes on a deterministic value.

The *Probabilistic Invariant* preserves a set of probabilities throughout loop execution.

Deterministic Invariant

Rabbit Hunting

while $i < n$ **do**

$rabbit := \texttt{UNIFORM}(k)$
$hunter := \texttt{UNIFORM}(k)$
$caught := caught \vee (hunter = rabbit)$
$i := i + 1$

end while

Deterministic Invariant

Rabbit Hunting

while $i < n$ **do**

$\{\exists m \leq n : Pr(i = m) = 1 \wedge Pr(i < n) = 1\}$

$rabbit$:= $\texttt{UNIFORM}(k)$
$hunter$:= $\texttt{UNIFORM}(k)$
$caught$:= $caught \vee (hunter = rabbit)$
i := $i + 1$

end while

Deterministic Invariant

Rabbit Hunting

while $i < n$ **do**

$\{\exists m \leq n : Pr(i = m) = 1 \land Pr(i < n) = 1\} \rightarrow$

$\{\exists m \leq n : Pr(i + 1 = m) = 1\}$

$rabbit$:= $\texttt{UNIFORM}(k)$

$hunter$:= $\texttt{UNIFORM}(k)$

$caught$:= $caught \lor (hunter = rabbit)$

i := $i + 1$

end while

Deterministic Invariant

Rabbit Hunting

while $i < n$ **do**

$\{\exists m \leq n : Pr(i = m) = 1 \wedge Pr(i < n) = 1\} \rightarrow$

$\{\exists m \leq n : Pr(i + 1 = m) = 1\}$

$rabbit$:= $\texttt{UNIFORM}(k)$

$hunter$:= $\texttt{UNIFORM}(k)$

$caught$:= $caught \vee (hunter = rabbit)$

i := $i + 1$

$\{\exists m \leq n : Pr(i = m) = 1\}$

end while

Probabilistic Invariant

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while $i < n$ **do**

$rabbit := \texttt{UNIFORM}(k)$
$hunter := \texttt{UNIFORM}(k)$

$caught := caught \vee (hunter = rabbit)$
$i := i + 1$

end while

Probabilistic Invariant

Rabbit Hunting

while $i < n$ **do**

$\{Pr(\neg caught) = \left(\frac{k-1}{k}\right)^i\}$

$rabbit$ `:=` `UNIFORM`(k)

$hunter$ `:=` `UNIFORM`(k)

$caught$ `:=` $caught \vee (hunter = rabbit)$

i `:=` $i + 1$

end while

Probabilistic Invariant

Rabbit Hunting

while $i < n$ **do**

$\{Pr(\neg caught) = \left(\frac{k-1}{k}\right)^i\}$

$rabbit := \mathtt{UNIFORM}(k)$

$hunter := \mathtt{UNIFORM}(k)$

$\{Pr(\neg caught \wedge hunter \neq rabbit) = \left(\frac{k-1}{k}\right)\left(\frac{k-1}{k}\right)^i\}$

$caught := caught \vee (hunter = rabbit)$

$i := i + 1$

end while

Probabilistic Invariant

Rabbit Hunting

while $i < n$ **do**

$\{Pr(\neg caught) = \left(\frac{k-1}{k}\right)^{i}\}$

$rabbit$:= `UNIFORM`(k)

$hunter$:= `UNIFORM`(k)

$\{Pr(\neg caught \wedge hunter \neq rabbit) = \left(\frac{k-1}{k}\right)^{i+1}\}$

$caught$:= $caught \vee (hunter = rabbit)$

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Probabilistic Invariant

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$caught := caught \vee (hunter = rabbit)$

$i := i + 1$

$\{Pr(\neg caught) = \left(\frac{k-1}{k}\right)^{i}\}$

end while

Catching Rabbits

Rabbit Hunting

$\{Pr(True) = 1\}$
i `:=` 0
$caught$ `:=` `F`

while $i < n$ **do**
    $rabbit$ `:=` $\texttt{UNIFORM}(k)$
    $hunter$ `:=` $\texttt{UNIFORM}(k)$
    $caught$ `:=` $(hunter = rabbit) \vee caught$
    i `:=` $i + 1$
end while

Catching Rabbits

Rabbit Hunting

$\{Pr(True) = 1\}$
i := 0
$caught$:= F
$\{Pr(\neg caught) = 1 \wedge Pr(i = 0) = 1\}$

while $i < n$ **do**
  $rabbit$:= UNIFORM(k)
  $hunter$:= UNIFORM(k)
  $caught$:= $(hunter = rabbit) \vee caught$
  i := $i + 1$
end while

Catching Rabbits

Rabbit Hunting

$\{Pr(True) = 1\}$
$i := 0$
$caught := \texttt{F}$
$\{Pr(\neg caught) = 1 \land Pr(i = 0) = 1\} \rightarrow$
$\{Pr(\neg caught) = \left(\frac{k-1}{k}\right)^i \land \exists m \leq n : Pr(i = m) = 1\}$
while $i < n$ **do**
  $rabbit := \texttt{UNIFORM}(k)$
  $hunter := \texttt{UNIFORM}(k)$
  $caught := (hunter = rabbit) \lor caught$
  $i := i + 1$
end while

Catching Rabbits

Rabbit Hunting

$\{Pr(True) = 1\}$
i := 0
$caught$:= F
$\{Pr(\neg caught) = 1 \wedge Pr(i = 0) = 1\} \rightarrow$
$\{Pr(\neg caught) = \left(\frac{k-1}{k}\right)^i \wedge \exists m \leq n : Pr(i = m) = 1\}$
while $i < n$ **do**
    $rabbit$:= UNIFORM(k)
    $hunter$:= UNIFORM(k)
    $caught$:= $(hunter = rabbit) \vee caught$
    i := $i + 1$
end while
$\{Pr(\neg caught) = \left(\frac{k-1}{k}\right)^i \wedge \exists m \leq n : Pr(i = m) = 1 \wedge i \not< n\}$

Catching Rabbits

Rabbit Hunting

$\{Pr(True) = 1\}$
i := 0
$caught$:= F
$\{Pr(\neg caught) = 1 \wedge Pr(i = 0) = 1\} \rightarrow$
$\{Pr(\neg caught) = \left(\frac{k-1}{k}\right)^i \wedge \exists m \leq n : Pr(i = m) = 1\}$
while $i < n$ **do**
    $rabbit$:= UNIFORM(k)
    $hunter$:= UNIFORM(k)
    $caught$:= $(hunter = rabbit) \vee caught$
    i := $i + 1$
end while
$\{Pr(\neg caught) = \left(\frac{k-1}{k}\right)^i \wedge Pr(i = n) = 1\}$

Catching Rabbits

Rabbit Hunting

$\{Pr(True) = 1\}$
$i := 0$
$caught := \mathtt{F}$
$\{Pr(\neg caught) = 1 \wedge Pr(i = 0) = 1\} \rightarrow$
$\{Pr(\neg caught) = \left(\frac{k-1}{k}\right)^i \wedge \exists m \leq n : Pr(i = m) = 1\}$
while $i < n$ **do**
  $rabbit := \mathtt{UNIFORM}(k)$
  $hunter := \mathtt{UNIFORM}(k)$
  $caught := (hunter = rabbit) \vee caught$
  $i := i + 1$
end while
$\{Pr(\neg caught) = \left(\frac{k-1}{k}\right)^i \wedge Pr(i = n) = 1\} \rightarrow$
$\{Pr(caught) = 1 - \left(\frac{k-1}{k}\right)^n\}$

Probabilistic Termination

What about programs that terminate probabilistically?

What about programs that terminate probabilistically?

$$\{\ \Pr(\text{True}) = 1\ \}$$

$$y \;\texttt{:=}\; \texttt{toss}\left(\tfrac{1}{2}\right);$$

$$\texttt{if}\ y\ \texttt{then}\ x \;\texttt{:=}\; 4\ \texttt{else}\ \texttt{loop}$$

$$\{\ \Pr(\text{x} = 4) = ?\ \}$$

Soundness

Theorem

All of the VPHL rules are sound with respect to the semantics of PrImp.

Verified

https://github.com/rnrand/VPHL

$\mathcal{FIN}$

Thank You Questions?

https://github.com/rnrand/VPHL